Predicates (Textbook §3.6)

A *predicate* is a *proposition* whose truth depends on the values of variables

$$P(n) ::= \text{"}n \text{ is a perfect square"}$$

- If P is a unary predicate, $P(n)$, being a proposition, is either true or false.
- A k-ary predicate takes k variables, e.g.,

  $$Py(x, y, z) ::= \text{"}x^2 + y^2 = z^2 \text{ for integers } x, y \text{ and } z\text{"}$$

  is a ternary predicate that characterizes Pythagorean triples. x, y and z are said to be *free variables* in the formula defining Py.

Satisfiability and Validity

These terms have the same high level meaning as in the propositional case.

Satisfiable: There is at least one value for each variable such that the formula is true

- Related to *existential* quantification
- Example: Are there values of x, y and z such that $Py(x, y, z)$?

Valid: The formula is true for all possible values of variables

- Related to *universal* quantification.

$$\text{Example:} Q(x, y) ::= \text{“if } y > 0 \text{ then } x + y > x\text{”}$$

Is this always true?

Quantifiers

Universal Quantifier: $\forall$ "for all"

$$\forall x \in \mathbb{R} \;\; x^2 \geq 0$$

Existential Quantifier: $\exists$ "there exists"

$$\exists x \in \mathbb{R} \;\; 5x^2 - 7 = 0$$

The sets may not be explicitly specified if they are obvious from context.

If multiple variables belong to the same domain, we may abbreviate:

Free Vs Bound Variables

- A quantifier "captures" or "binds" a variable. It is no longer free to take any possible value

$$P(n) ::= \exists x \in \mathbb{N},\ n = x^2$$

  - This formula has two variables n and x, but x has been bound by the quantifier.
  - n continues to be *free:* It can take any value.
    - Some values of n make $P(n)$ true, while others render it false.

Examples of Quantified Statements

“Every American has a dream.”

Examples of Quantified Statements

Every even integer greater than 2 is the sum of two primes

- For every even integer n greater than 2, there exist primes p and q s.t. $n = p + q$

Order of Quantifiers

- Sometimes the order does not matter
  - specifically, when the same quantifier is nested
  - Examples

$$\forall x\ \forall y\ [x \geq y \vee y > x]$$
$$\exists x\ \exists y\ \exists z\ [(x + 2y = z) \wedge (x - y = z)]$$

- But other times, they mean a lot!
  - Specifically, if the order of dissimilar quantifiers is changed, that can change the meaning of the formula
  - Example:

$(\forall x\ \exists y\ y > x)$ is not the same as $(\exists y\ \forall x\ y > x)$

Negating Quantifiers

- Not everyone likes ice cream.
- There is someone who does not like ice cream
- There is no one who likes being mocked
- Everyone dislikes being mocked
- $\forall$ is equivalent to $\neg \exists \neg$
- $\exists$ is equivalent to $\neg \forall \neg$

Unit Summary

- Predicate Vs Proposition
- Satisfiability and Validity
- Quantifiers
  - Free and Bound variables
  - Conversion between English and Logical Formulas
- Order of quantifiers
- Negating quantifiers