Sets (Textbook §4.1)

Informally, a set is a collection of things:

$A = \{Alex, Tippy, Shells, Shadow\}$	Pet names
$B = \{red, blue, yellow\}$	Primary colors
$C = \{\{a, b\}, \{b, c\}, \{a, d\}, \{a, b, c\}\}$	A set of sets

- The order of elements is irrelevant — a set is an unordered collection
- There are no repeated elements in a set

Infinite Sets and Set Builder Notation

Here is an example of an infinite set:

$$T ::= \{0, 3, 6, 9, 12, \ldots\}$$

We can state this more precisely as

$$T ::= \{x \in \mathbb{N} | x \text{ is a multiple of } 3\}$$

or more compactly as:

$$T ::= \{3n | n \in \mathbb{N}\}$$

Predefined Sets

$\emptyset$	The empty set
$\mathbb{N}$	The set of natural numbers, i.e., $\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}$	The set of integers, i.e., $\{0, -1, 1, -2, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of Complex numbers

Set Operators

Operation	Operator	Definition/Example
"belongs to"	$\in$	$2 \in \mathbb{N}, \sqrt{5} \notin \mathbb{Q}$
Union	$\cup$	$A \cup B = \{x \vert x \in A \ \underline{\text{or}} \ x \in B\}$
Intersection	$\cap$	$A \cap B = \{x \vert x \in A \ \underline{\text{and}} \ x \in B\}$
Difference	$-$	$A - B = \{x \vert x \in A \ \underline{\text{and}} \ x \notin B\}$
Subset	$\subseteq$	$A \subseteq B$ iff for all $x \in A, \ x \in B$
Proper Subset	$\subset$	$A \subset B$ iff $A \subseteq B$ and $A \neq B$

Showing Two Sets A and B are Equal

- Show that $x \in A$ implies $x \in B$ and vice-versa
- Show $A \subseteq B$ and $B \subseteq A$

Universal Set and Complement

- If all the sets being considered are a subset of a larger set U, we call that a *universal* set
- In the presence of a universal set, we can define a *complement* $\overline{A}$ as follows:

  $$\overline{A} = U - A$$

  Alternatively:

  $$\overline{A} = \{x \in U | x \notin A\}$$

Some properties of Set Operators

- $\cup$ and $\cap$ are commutative and associative
  - Follows from the definition of these operators, and the fact that the boolean connectives $\underline{\text{or}}$ and $\underline{\text{and}}$ are both associative and commutative
- $\cup$ distributes over $\cap$ and vice-versa

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

  - Can be established from the definition of $\cup$ and $\cap$
- De Morgan's Law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Power Set

Given a set A, the *power set* of A, denoted $\wp(A)$ is

$$\wp(A) = \{x | x \subseteq A\}$$

- A powerset always includes $\emptyset$
- $\wp(A)$ always includes A

Size of Power Set

If the size of A, denoted $|A|$, is n, what is the size of $\wp(A)$?

Products and Tuples

Cartesian Product of sets A and B: $A \times B = \{(a, b) | a \in A, b \in B\}$

Example: Let $A = \{1, 4, 9\}$, $B = \{a, e, i, o, u\}$

$$\begin{aligned} A \times B = \{ \; & (1, a), (1, e), (1, i), (1, o), (1, u), \\ & (4, a), (4, e), (4, i), (4, o), (4, u), \\ & (9, a), (9, e), (9, i), (9, o), (9, u)\} \end{aligned}$$

Products and Tuples

- (x, y) is called an (ordered) *pair*
- (x, y, z) is a *triple*, or a 3-way product
- More generally, $(x_1, x_2, \ldots, x_n)$ is an n-tuple, or just a *tuple*
  - It is a cartesian product on n sets, denoted $\times_n$
- *Unlike sets, the order of components is important in a tuple.* $A \times B \neq B \times A$
- A^n is short hand for an n-way product

$$\underbrace{\times_n(A, A, \ldots A)}_{n \text{ times}}$$

  - Technically, $A \times (A \times A) \neq \times_3(A, A, A)$

$$A \times (A \times A) = \{(a, (b, c)) | a, b, c \in A\}$$
$$\times_3(A, A, A) = \{(a, b, c) | a, b, c \in A\}$$

Sets: Summary

- Definition of sets
- Set builder notation
- Set operators: membership, subset, union, intersection, difference
  - Properties of set operators: commutativity, associativity, distributivity
- Set equality
- Universal set, set complement and De Morgan's Laws
- Power Set
- Cartesian Product and Tuples