Series and Summations (Textbook §13.1 to §13.5)

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- Often, we need to find closed form solutions of a series
  - Applications arise in algorithm analysis, data analysis, financial applications, etc.
- Examples

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

- Sometimes, we are interested in products, and in reasonable approximations

Sterling's approximation: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

  - Interestingly, π and e appear in what is ultimately an integer!
  - Approximation error is just 1% for $n = 6$, decreasing to 0.1% at $n = 100$

Prize Payout Question

- You won the a large cash award. (Congratulations!)
- You are given the option of either one large payout of \$20M, or annual payments of \$1M per year for ever.
- Which one should you take?

Future Value of Money

- To make an informed decision, you need to consider:
  - the interest your lump sum payment will earn
    - Let us assume you can earn 6% with very safe investments
- Compare what you will have after, say, 100 years in each case

Alternative: Current Value of Future Money

- Key idea: \$1M you receive next year is worth only \$1/1.06M worth today
  - Reason: At 6% interest, \$1/1.06 will become $\$1/1.06 * (1 + 0.06) = 1M$ next year
- So, current value of all monies you will get

$$\begin{aligned}
&= 1 + 1/1.06 + 1/1.06^2 + 1/1.06^3 + \cdots \\
&= \sum_{i=0}^{\infty} x^i \text{ where } x = \frac{1}{1.06} \\
&= \frac{1 - x^{\infty}}{1 - x} \text{ using the formula for geometric series} \\
&= \frac{1 - \left(\frac{1}{1.06}\right)^{\infty}}{1 - \frac{1}{1.06}} = \frac{1}{1 - \frac{1}{1.06}} \\
&= 17.7M
\end{aligned}$$

Current Price of Future Cash Flow

- This is how annuities are priced
  - Retirees often purchase annuities using part (or all) of their retirement savings
  - Financial institutions calculate the price to charge using a calculation similar to the above
    - Annuities are paid only while the purchaser is alive
    - Modify calculation to use finite rather than infinite sum
- Pensions are also calculated in a similar way

Annuity Based on Expected Lifetime

- Let us use the above calculation to price a 20-year annuity:

$$
\begin{aligned}
&= 1 + 1/1.06 + 1/1.06^2 + 1/1.06^3 + \cdots 1/1.06^{19} \\
&= \sum_{i=0}^{19} x^i \text{ where } x = \frac{1}{1.06} \\
&= \frac{1 - x^{20}}{1 - x} \text{ using the formula for geometric series} \\
&= \frac{1 - \left(\frac{1}{1.06}\right)^{20}}{1 - \frac{1}{1.06}} \\
&= 12.16M
\end{aligned}
$$

Summation Techniques: Perturbation Method

- Find a “perturbation” that makes all terms identical:
- Let $S = 1 + 2 + 3 + \cdots + n$
- Create another instance of S by reversing the order of terms

$$\begin{array}{rcccccccccc}
S & = & 1 & + & 2 & + & 3 & + & \cdots & + & n \\
S & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \\
\hline
2S & = & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1)
\end{array}$$

- Simplifying, we get $S = n(n+1)/2$

Summation Techniques: Perturbation Method

- Find a “perturbation” that can cancel out most terms:
- Let $S = 1 + x + x^2 + \cdots + x^n$
- Now, $xS = \quad x + x^2 + \cdots + x^n + x^{n+1}$
- Subtract one from the other:

$$xS - S = x^{n+1} - 1$$

- Simplifying, we get:

$$S = \frac{x^{n+1} - 1}{x - 1}$$

Voila! We have derived the formula for sum of geomeric series!

Alternative Perturbation for Arithmetic Progression

- Sometimes, you perturb a seemingly unrelated sequence in order to get your sum
- Start with sum of squares, and perturb to cancel out most terms:

$$\begin{array}{cccccccccccccc}
\sum_{i=1}^n & i^2 & = & & & 1^2 & + & 2^2 & + & \cdots & + & (n-1)^2 & + & n^2 \\
\sum_{i=1}^n & (i-1)^2 & = & 0^2 & + & 1^2 & + & 2^2 & + & \cdots & + & (n-1)^2 & & \\
\hline
\sum_{i=1}^n & (i^2-(i-1)^2) & = & 0 & + & 0 & + & 0 & + & \cdots & + & 0 & + & n^2
\end{array}$$

- Simplifying lhs using the identity $a^2 - b^2 = (a-b)(a+b)$, we get

$$\sum_{i=1}^{n} i^2 - (i-1)^2 = \sum_{i=1}^{n} (i-(i-1))(i+i-1) = \sum_{i=1}^{n} 2i - 1 = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2\left(\sum_{i=1}^{n} i\right) - n$$

- Setting this equal to the rhs and then simplifying, we get: $2(\sum_{i=1}^n i) - n = n^2$
- i.e., $\sum_{i=1}^n i = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

Summation of i^k

- For any $\sum i^k$, you can use the same method! Let us examine $\sum i^2$

$$\begin{array}{rcccccccccccc}
\sum_{i=1}^n & i^3 & = & & & 1^3 & + & 2^3 & + & \cdots & + & (n-1)^3 & + & n^3 \\
\sum_{i=1}^n & (i-1)^3 & = & 0^3 & + & 1^3 & + & 2^3 & + & \cdots & + & (n-1)^3 & & \\
\hline
\sum_{i=1}^n & (i^3-(i-1)^3) & = & & & & & & & \cdots & & & & n^3
\end{array}$$

- Simplifying lhs using the identity $a^3 - b^3 = (a-b)(a^2+b^2+ab)$, we get

$$\begin{aligned}
\sum_{i=1}^n i^3-(i-1)^3 &= \sum_{i=1}^n (i-(i-1))(i^2+(i-1)^2+i(i-1)) \\
&= \sum_{i=1}^n 3i^2-3i+1 \\
&= 3\sum_{i=1}^n i^2 - 3\sum_{i=1}^n i + \sum_{i=1}^n 1 \qquad = n^3
\end{aligned}$$

- Further simplifying, $\sum_{i=1}^n i^2 = (n^3 - n + 3\sum_{i=1}^n i)/3$
- Substituting for $\sum_{i=1}^n i$ from previous slide into rhs and simplifying:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Perturbing By Differentiation: Arithmetico Geometric Prog.

- How do you find the following sum?

$$\sum_{i=1}^{n-1} ix^i = x + 2x^2 + 3x^3 + \cdots (n-1)x^{n-1}$$

- This looks kind of similar to geometric progression, so start with that:

$$\sum_{i=1}^{n-1} x^i = x + x^2 + x^3 + \cdots x^{n-1} = \frac{1-x^n}{1-x}$$

- Each term ix^i in AGP seems like it's obtained by differentiating the term x^{i+1} in GP!

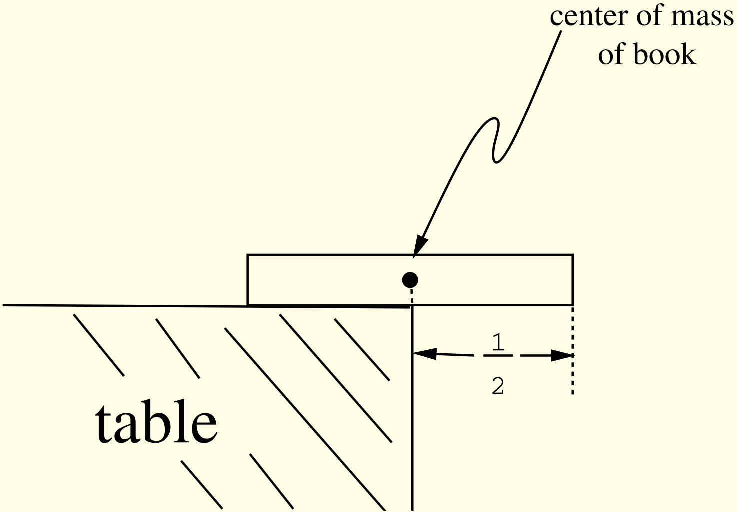
Perturbing By Differentiation: Arithmetico Geometric Prog.

$$\begin{aligned}
\frac{d}{dx}\left(\sum_{i=1}^{n-1} x^i\right) &= \frac{d}{dx}\left(\frac{1-x^n}{1-x}\right) \\
\sum_{i=1}^{n-1} i x^{i-1} &= \frac{-nx^{n-1}(1-x)-(-1)(1-x^n)}{(1-x)^2} \\
&= \frac{1-nx^{n-1}+(n-1)x^n}{(1-x)^2} \\
&= \frac{(n-1)x^n-nx^{n-1}+1}{(1-x)^2}
\end{aligned}$$

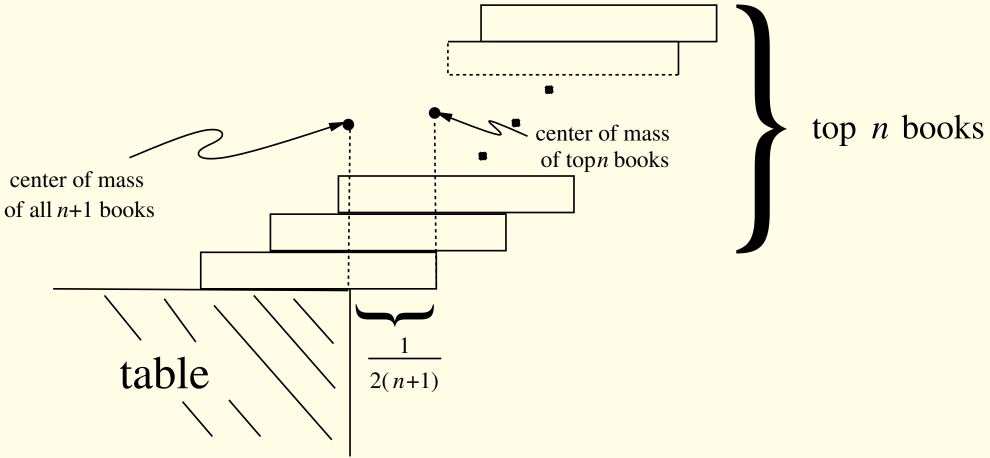
The AGP we want is not exactly the lhs here. But if we multiply both sides by x, we will be there:

$$\sum_{i=1}^{n-1} i x^i = \frac{(n-1)x^{n+1}-nx^n+x}{(1-x)^2}$$

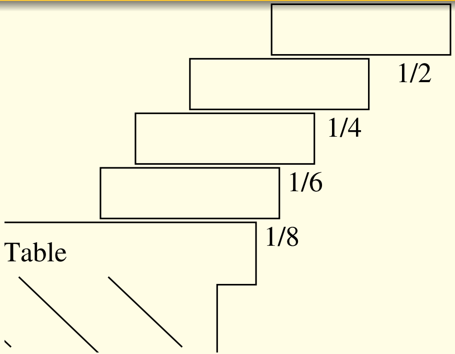
Hanging Blocks



How much overhang do you get with *n*th block?



Hanging Blocks: The Final Picture



The summation we need: $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i}$

The sum $\sum_{i=1}^{n} \frac{1}{i}$ is called the *n^{th} Harmonic number* H_n. But how do we compute it?

Integration: The Master Technique for Approximating Sums

- Consider

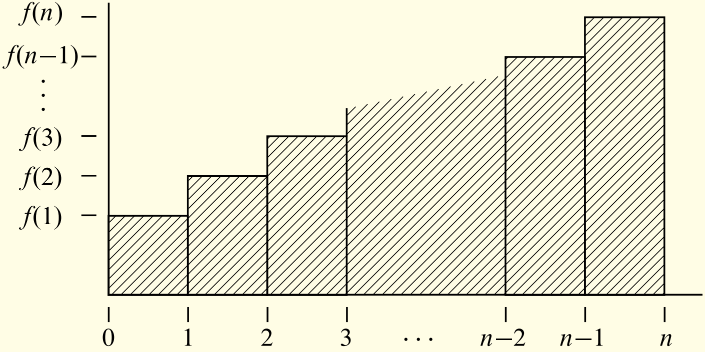
$$\sum_{i=1}^{n} \frac{1}{i}$$

  - We can't use any tricks here — in fact, no closed from expression is known for this summation.

- *Idea:* Approximate using integration
  - Integral $\int_1^n f(x)\,dx$ yields the *area under* the curve $f(x)$ between $x = 1$ and $x = n$
    - Can we relate it to the discrete sum $\sum_{x=1}^{n} f(x)$?

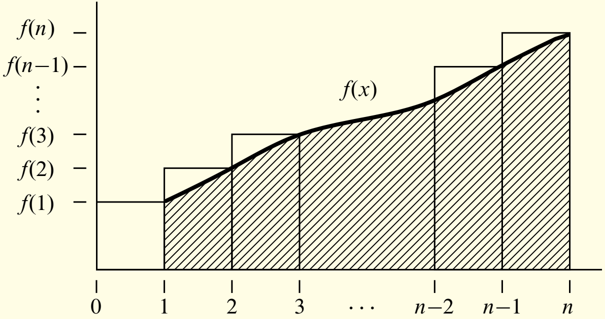
Pictorial representation of Discrete Sum

- The discrete sum represents the area of the shaded region:



Pictorial Comparison of Discrete Sum and Integral

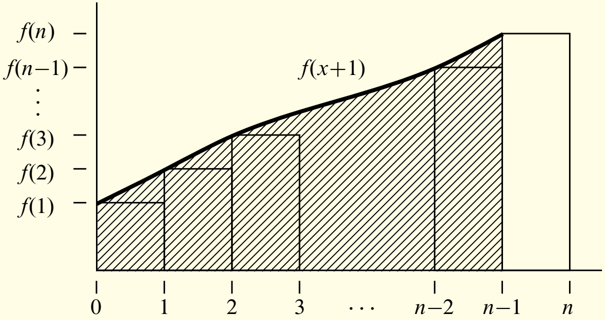
- The integral $\int_1^n f(x)\,dx$ represents the area of the shaded region:



- Unshaded region within the boxes represents the difference between the integral and discrete sum $\sum_{x=1}^{n} f(x)$
- So, $\sum_{x=1}^{n} f(x) \approx f(1) + \int_1^n f(x)\,dx$ is an under-approximation of the discrete sum.

Pictorial Comparison of Discrete Sum and Integral

- Let us shift f one unit to the left, i.e., plot $f(x+1)$ instead.



- $\sum_{x=1}^{n} f(x) \approx f(n) + \int_{1}^{n} f(x)\, dx$ is an over-approximation in this case.

Dotting the i's ...

Weakly Increasing Functions

A function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is *weakly increasing* iff $x < y \rightarrow f(x) \leq f(y)$.

Summation By Integration

Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, and let S and I be defined as follows:

$$S ::= \sum_{i=1}^{n} f(i) \qquad\qquad I ::= \int_{x=1}^{n} f(x)dx$$

- If f is weakly increasing then $I + f(1) \;\leq\; S \;\leq\; I + f(n)$

Dotting the i's ...

Weakly Increasing and Decreasing Functions

A function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is *weakly increasing* iff $x < y \rightarrow f(x) \leq f(y)$.

A function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is *weakly decreasing* iff $x < y \rightarrow f(x) \geq f(y)$.

Summation By Integration

Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, and let S and I be defined as follows:

$$S ::= \sum_{i=1}^{n} f(i) \qquad\qquad I ::= \int_{x=1}^{n} f(x)dx$$

- If f is weakly increasing then $I + f(1) \;\leq\; S \;\leq\; I + f(n)$
- If f is weakly decreasing then $I + f(1) \;\geq\; S \;\geq\; I + f(n)$

Back to the hangling blocks example ...

We need to integrate $1/x$. Specifically:

$$\int_1^n \frac{1}{x}\,dx = \ln(x)\Big|_1^n = \ln(n) - \cancel{\ln(1)}$$

Noting that $f(1) = 1$ and $f(n) = 1/n$, we have the bound[1]:

$$\frac{1}{n} + \ln(n) \le \sum_{i=1}^{n} \frac{1}{i} \le \ln(n) + 1$$

- The maximum overhang is infinite!
- We get overhang longer than one full block when $n = 4$

[1] Since $1/x$ is a decreasing function, so we need to use the bound $I + f(n) \ \le\ S \ \le\ I + f(1)$

Approximating Factorial ...

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n = \prod_{i=1}^{n} i$$

Can we turn this into a summation?

$$\ln n! = \ln 1 + \ln 2 + \cdots \ln n = \sum_{i=0}^{n} \ln i$$

Now we can apply our integration trick! But first we need to integrate $\ln x$:

$$\int_1^n \ln(x) dx = x \ln(x) - x \Big|_1^n = n \ln(n) - n - \cancel{1 \ln(1)} + 1 = n \ln(n) - n + 1$$

Approximating Factorial ...

Using the formula for approximating sums using integration,

$$\cancel{\ln(1)} + \underline{n\ln(n) - n + 1} \leq \sum_{i=0}^{n} \ln(i) \leq \underline{n\ln(n) - n + 1} + \ln(n)$$

Let us take the average of the two bounds as our estimate:$(n + 0.5)\ln(n) - n + 1$

Now, take the exponent of every term so as to get rid of the ln operations.

Our result: $n! = \frac{n^{n+0.5}}{e^{n-1}} = e\sqrt{n}\left(\frac{n}{e}\right)^n$

Sterling's approx: $n! = \sqrt{2\pi}\sqrt{n}\left(\frac{n}{e}\right)^n$

Summations and Runtimes of Algorithms

```
{ x for x in 2..n
      if forall (y in 2..(x-1)) x % y != 0 }
```

- What is its runtime, as a function of n
- Note: If you replace n with an integer that is not too large, this is a valid DML program.

More efficient computation of primes

- If x is nonprime, then $x = y \times z$ where $y, z > 1$.
- This means that both y and z cannot be greater than $\sqrt{x}$
  - Or else, $y \times z > \sqrt{x} \times \sqrt{x} = x$
- So, the inner loop can stop at $\sqrt{x}$ instead of $x - 1$

```
{ x for x in 2..n
      if forall (y in 2..floor(sqrt(x))) x % y != 0 }
```

- What is the runtime of this program?

Sum of square roots using integrals

- Integral:

$$\int_1^n \sqrt{x}\,dx = \frac{x^{3/2}\Big|_1^n}{3/2} = \frac{2}{3}(n^{3/2}-1)$$

- So, the actual value is bounded between the under and over approximations:

$$1 + \frac{2}{3}(n^{3/2}-1) \le \sum_{x=1}^{n} \sqrt{x} \le \sqrt{n} + \frac{2}{3}(n^{3/2}-1)$$

  (Note: The square root function is weakly increasing.)

- For larger n values, $n^{3/2}$ dominates over $\sqrt{n}$, so the approximation is pretty good.
  - e.g., the error is about $1/n$,
  - i.e. about 10% for $n \ge 10$, about 1% for $n \ge 100$, etc.

Summary of Summation Techniques

- Perturbation Method
  - Example 1: Geometric Progression: $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$
  - Example 2: Arithmetic Progression: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  - Example 3: Sum of i^k: $\sum_{i=1}^{2} = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^{3} = \left(\frac{n(n+1)}{2}\right)^2$
- Using differentiation
  - Example 4: Arithmetico Geometric Progression: $\sum_{i=1}^{n-1} ix^{i-1} = \frac{(n-1)x^n - nx^{n-1}+1}{(1-x)^2}$
- Using integration
  - Example 5: $1 + \frac{2}{3}(n^{3/2} - 1) \leq \boxed{\sum_{x=1}^{n} \sqrt{x}} \leq \sqrt{n} + \frac{2}{3}(n^{3/2} - 1)$
  - Example 6: Factorial: $n \ln(n) - n + 1 \leq \boxed{\sum_{i=0}^{n} \ln(i)} \leq n \ln(n) - n + 1 + \ln(n)$
  - Example 7: Hanging blocks: $\frac{1}{n} + \ln(n) \leq \boxed{\sum_{i=1}^{n} \frac{1}{i}} \leq \ln(n) + 1$

A point about in-class mathematical derivations ...

- For many math derivations, you need to work them out yourself in order to fully understand every step.
  - There is absolutely no reason to feel discouraged if you don't follow every step of a derivation in class
- But I do want you to understand the steps to a degree that you feel you can fill in any gaps offline.
  - Or else, stop and ask questions.
- Often, the main challenge students face is that they may not have understood some of the basic concepts in math as clearly as they thought.
  - We need to make a concerted and concious effort to understand these gaps and improve our understanding in order to do well in college.